

## Saturation of the production of quantum entanglement between weakly coupled mapping systems in a strongly chaotic region

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The production of quantum entanglement between weakly coupled mapping systems, whose classical counterparts are both strongly chaotic, is investigated. In the weak-coupling regime, it is shown that time-correlation functions of the unperturbed systems determine the entanglement production. In particular, we elucidate that the increment of the nonlinear parameter of coupled kicked tops does not accelerate the entanglement production in the strongly chaotic region. An approach to the dynamical inhibition of entanglement is suggested.

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In a quantum composite system, even if the subsystems are remotely separated and the whole system is in a pure state, the subsystems generically have a nonclassical correlation [1]. This striking phenomenon is called quantum entanglement [2], which is utilized not only to achieve the procedures that have no classical analogs (e.g., quantum information processing [3]), but also to realize the “classical world” in which quantum interference phenomena are “decohered” as a result of quantum dynamics [4]. Even when there is no quantum entanglement between subsystems, a weak interaction between the subsystems generally produces quantum entanglement during unitary time evolutions [5]. This is an important dynamical origin of decoherence [4].

Through a number of numerical experiments, it is known that the productions of entanglements induced by quantum dynamics heavily depend on the qualitative nature of the corresponding classical dynamics, namely, regular or chaotic [6–9], as is easily expected from the studies of “quantum chaos” [10]. On one hand, in classically regular systems, the confinement of phase-space dynamics in a narrow region enclosed by KAM tori makes it difficult to produce strong entanglements, in general [11]. On the other hand, the absence of such dynamical barriers in classically chaotic systems promotes the production of entanglement. Although there are quantum effects on the phase-space dynamics, e.g., tunnelings and localizations [13], in both regular and chaotic systems, it is confirmed that the scenario above qualitatively holds [6–9].

This motivates the next question: In the chaotic region, does stronger chaos enhance the production of entanglement? Looking for an analogy of a study on quantum open systems [14], Miller and Sarkar obtained a numerical result which suggests that the linear instability of classical dynamics enhances the production of entanglement [9]. Their numerical experiment however concerns only in the *weakly* chaotic region where chaotic seas and tori coexist. The com-

plexity of phase-space dynamics is the source of the difficulty in obtaining a theoretical explanation for Miller and Sarkar’s result.

Our aim is to provide a theoretical argument of entanglement production in weakly coupled chaotic systems. In contrast to Miller and Sarkar’s work, we focus on the *strongly* chaotic region where the effect of tori is small, to facilitate in obtaining a theoretical explanation. Starting from separable pure states, we examine the productions of quantum entanglement due to unitary time evolutions. The entanglement production processes are slow due to the weak coupling. Furthermore, the recurrence time of classically chaotic systems is relatively long. Hence, the entanglement production processes are nearly stationary processes, at least, in a short time period. This enables us to introduce an entanglement production rate. We investigate how the entanglement production rate depends on the nonlinear parameter below.

Our numerical experiments employ coupled kicked tops [9]. First, we introduce their constituent, a kicked top [15], which is described by the Hamiltonian  $\hat{H}_k \equiv \pi \hat{J}_y / 2 + \Delta(t) k \hat{J}_z^2 / (2j)$ , where  $\hat{J}_i$  is the  $i$ th component of the angular momentum operator of the top,  $j$  is the magnitude of the angular momentum,  $k$  is a nonlinear parameter, and  $\Delta(t) \equiv \sum_{n \in \mathbb{Z}} \delta(t-n)$  is a “periodic delta function.” Second, we employ the following Hamiltonian to describe the whole system that is composed of two kicked tops:

$$\hat{H} \equiv \hat{H}_{k_1} \otimes \hat{1} + \hat{1} \otimes \hat{H}_{k_2} + \epsilon \hat{V} \Delta(t), \quad (1)$$

where  $\epsilon$  is a coupling constant,  $\hat{V} \equiv \hat{J}_{1z} \hat{J}_{2z} / j$  is the interaction Hamiltonian, and  $J_{iz}$  is the  $J_z$  of the  $i$ th top. We report the case where the magnitudes of the angular momenta of the two subsystems are the same value,  $j$ .

Since we focus on the case where the total system is in a pure state, our choice of a measure of quantum entanglement between the two subsystems, is the von Neumann entropy

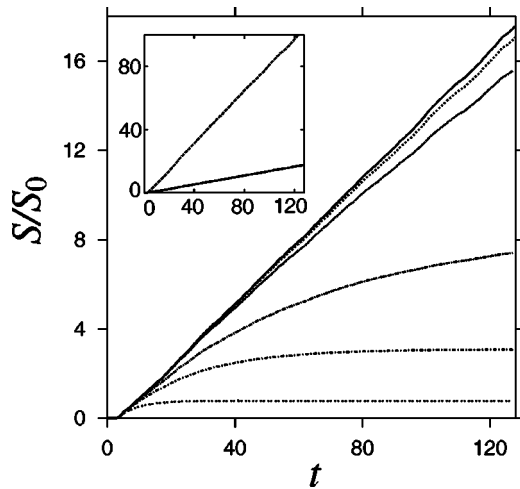


FIG. 1. Time evolutions of quantum entanglement, measured by the entropies of a subsystem. The entropies are scaled by  $S_0 = 2\epsilon^2 j^2$  [cf. Eq. (2)]. From bottom to top, we depict  $S_{\text{lin}}$  with  $\epsilon = 10^{-2}$ ,  $5 \times 10^{-3}$ ,  $3 \times 10^{-3}$ ,  $10^{-3}$ ,  $5 \times 10^{-4}$ ,  $10^{-4}$ . An estimation  $S_{\text{lin}}^{\text{PT}}$  by our perturbation formula (2) is degenerate with the case  $\epsilon = 10^{-4}$ . In the inset, the solid line and the dotted line correspond to  $S_{\text{lin}}$  and  $S_{\text{vN}}$ , respectively, at  $\epsilon = 10^{-4}$ . The values of the nonlinear parameters are  $k_1 = k_2 = 7.0$ , which means that the corresponding classical tops are strongly chaotic [15]. The magnitude of the angular momenta is chosen to be large  $j = 80$ , in order to investigate the semiclassical regime. The center of the initial wave packet, which is a product of spin-coherent states [17], is  $(\theta_1, \phi_1, \theta_2, \phi_2) = (0.89, 0.63, 0.89, 0.63)$ .

$S_{\text{vN}}$  of the first subsystem [16]. Note that the von Neumann entropy of the second subsystems is equal to that of the first subsystem, when the total system is in a pure state. At the same time, we employ the linear entropy  $S_{\text{lin}}$  instead of  $S_{\text{vN}}$ , to facilitate theoretical arguments. In our numerical experiments, these entropies behave qualitatively similarly.

We examine the productions of quantum entanglements, by observing  $S_{\text{vN}}$  and  $S_{\text{lin}}$ , during unitary time evolutions whose initial states are product (i.e., separable) states. A typical result in the strongly chaotic region is shown in Fig. 1: When the coupling constant  $\epsilon$  is small, there is a  $t$ -linear entanglement production region, which is wide enough to be called the “stationary” entanglement production. Note that during the stationary entanglement production, the state vectors of the subsystems are spread over the phase space of the subsystems (Fig. 2). In other words, the phase-space distribution of each subsystem is nearly in “equilibrium.” We accordingly expect that each subsystem plays a role of a chaotic “heat bath” for its partner [20]. The  $t$ -linear entanglement production region starts at a time step  $T'$ , after a short transient to attain the equilibrium of the phase space distribution of the subsystems, and ends at a time step  $T''$ , until the increment of the entropy reaches its equilibrium (see Fig. 1, larger  $\epsilon$ ).

In order to explain the  $t$ -linear, stationary entanglement production, we employ a time-dependent perturbation theory, whose small parameter is a coupling constant  $\epsilon$ , to evaluate the linear entropy  $S_{\text{lin}}(t)$  at  $t$ th step. The resultant formula is

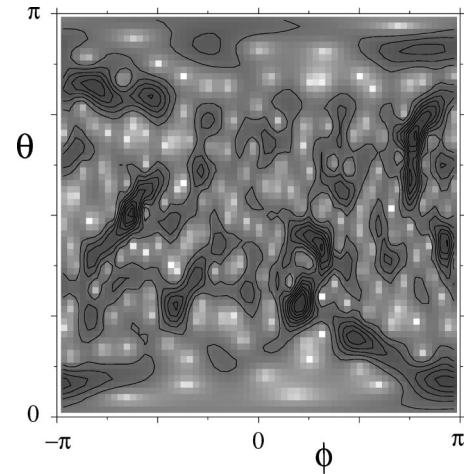


FIG. 2. The Husimi function [18] of the first subsystem at  $t = 15$ , during the stationary entanglement production region. The contour and density plots are in normal and logarithmic scales, respectively. Note that this region is beyond the Ehrenfest time [19]. We choose  $\epsilon = 10^{-4}$ . Other parameters are the same as in Fig. 1.

$$S_{\text{lin}}^{\text{PT}}(t) = S_0 \sum_{m=1}^t \sum_{n=1}^t D(m, n), \quad (2)$$

where  $S_0 \equiv 2\epsilon^2 j^2$  and  $D(m, n)$  is a time-correlation function of the uncoupled system. Furthermore, since the interaction Hamiltonian  $\hat{V}$  takes a bilinear form,  $D(m, n)$  is decomposed as follows

$$D(m, n) \equiv C_1(m, n) C_2(m, n), \quad (3)$$

where  $C_i(m, n) \equiv j^{-2} (\langle \hat{J}_{iz}^m \hat{J}_{iz}^n \rangle - \langle \hat{J}_{iz}^m \rangle \langle \hat{J}_{iz}^n \rangle)$  is a normalized correlation function of  $\hat{J}_{iz}^n$ , which is evolved by the unperturbed Hamiltonian  $H|_{\epsilon=0}$  until the  $n$ th step, with an initial condition  $\hat{J}_{iz}^0 = \hat{J}_{iz}$ , and the expectation value  $\langle \cdot \rangle$  is respect to the unperturbed system. The details for obtaining the formula (2) will be shown elsewhere [21].

We remark on the entanglement production formula (2).

(i) Although we start from the evaluation of the entropy of a subsystem, the formula (2) is in a symmetric form with respect to the exchange of the two subsystems. This is consistent with the symmetric nature of quantum entanglement when the whole system is in a pure state.

(ii) Since our approach does not take into account the effect of the recurrence, the formula (2) would have qualitatively different applicabilities to the classically regular and chaotic systems. On one hand, for classically regular systems, our theory would break down in a relatively short time period, due to the smallness of the period of the recurrence. On the other hand, for chaotic systems, we numerically confirmed that our theory works for rather long time periods.

(iii) Our formula has a similarity with those in phenomenological descriptions of linear irreversible processes [22], in the sense that these theories use time-correlation functions to describe relaxation phenomena. This is useful both for

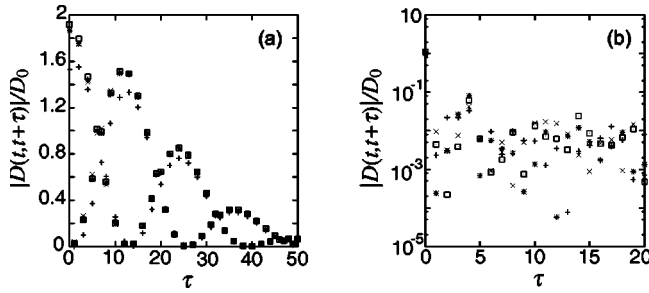


FIG. 3. The  $\tau$  dependence of the correlation function  $|D(t + \tau, t)|$  for (a) a regular system ( $k_1 = k_2 = 1.0$ ) and (b) a chaotic system ( $k_1 = k_2 = 7.0$ ) with  $j = 80$ . Different symbols correspond to different values of  $t$  ( $+$ ,  $t = 40$ ;  $\times$ ,  $t = 50$ ;  $*$ ,  $t = 60$ ;  $\square$ ,  $t = 70$ ). Note that (b) employs a normal-log scale.

discussing phenomenological arguments and for making a link with a phenomenological theory and a microscopic theory [22].

Before applying the formula (2) numerically, let us confirm that the time-correlation function  $D(m, n)$ , which is the most important ingredient of the formula (2), strongly depends on the dynamics of the classical counterparts (Fig. 3). On one hand, in the regular case,  $D(m, n)$  decays slowly with large oscillations, as the time interval  $|m - n|$  becomes large. On the other hand, the chaotic dynamics makes the decay of  $D(m, n)$  much faster. Such a rapid convergence of the correlation function, together with the formula (2), implies the  $t$ -linear, stationary entanglement production region (see Fig. 1).

The perturbation formula (2) provides an approximate estimation of the entanglement production rate  $\Gamma$  of the  $t$ -linear, stationary entanglement production region:

$$\Gamma^{\text{PT}} \equiv S_0 \frac{1}{T'' - T'} \sum_{m=T'}^{T''} \sum_{n=T'}^{T''} D(m, n), \quad (4)$$

where  $T'$  and  $T''$  are the start and the end of the  $t$ -linear entanglement production region, respectively. In the strongly chaotic region, where the effect of tori is small, it is possible to give an analytical estimation for  $\Gamma^{\text{PT}}$ , Eq. (4). Since  $D(m, n)$  is the product of the fluctuations of  $J_{iz}$ , whose distribution functions become quickly uniform due to the chaotic dynamics (see Fig. 2), we assume that  $D(m, n)$  decays exponentially,

$$D(m, n) = D_0 \exp(-\gamma |m - n|). \quad (5)$$

The prefactor  $D_0$  is determined by the magnitude of the fluctuations of  $J_{1z}$  and  $J_{2z}$ , whose distribution functions are almost uniform in strongly chaotic systems (see Fig. 2). We accordingly assume  $D_0 = (1/3)^2$ , which is independent of  $k_1$  and  $k_2$ . Furthermore, it is natural to expect that the decay rate  $\gamma$  of  $D(m, n)$  increases as the degree of chaos of the classical counterpart becomes stronger (i.e., as the values of nonlinear parameters  $k_1$  and  $k_2$  increase), when the effect of tori is negligibly small, although we could not obtain a pre-

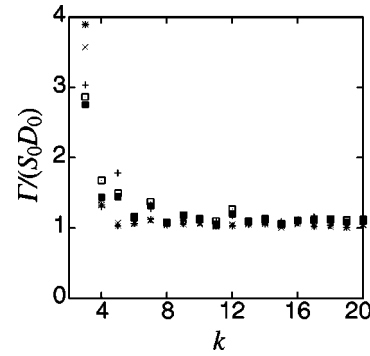


FIG. 4. Dependences of the entanglement production rates  $\Gamma$ , which is measured by linear entropy, on the nonlinear parameter  $k = k_1 = k_2$ . In order to show typical  $k$  dependences, we choose several initial conditions (depicted by different marks) that occur in the chaotic sea. Although the entanglement production heavily depends on the initial condition in the weakly chaotic region, the disappearance of tori weakens the initial condition dependence in the strongly chaotic region. Other parameters are the same as in Fig. 2.

cise nonlinear parameter dependence of  $\gamma$  due to the fact that the decay is very fast (within a step) in the strongly chaotic region [see Fig. 3(b)].

As long as the  $t$ -linear region is wide enough, i.e.,  $T'' - T' \gg \gamma^{-1}$ , the exponential decay assumption (5) provides an estimation

$$\Gamma^{\text{PT}} \simeq S_0 D_0 \coth(\gamma/2). \quad (6)$$

This provides our main result: in the limit that the classical counterpart is strongly chaotic, i.e., in the limit  $\gamma \rightarrow \infty$ , the perturbation theory (2) predicts that the entanglement production rate  $\Gamma$  converges to a finite value. Furthermore, the convergence is expected to be fast, when  $\gamma$  is larger than unity. Our main claim is that the effect of the enhancement of entanglement due to chaotic dynamics saturates in the strongly chaotic region, in contrast with the weakly chaotic systems [7–9]. This prediction is confirmed by our numerical experiments (Fig. 4). We explain the reason why the strongly chaotic systems have smaller entanglement production rates, in comparison with the weakly chaotic systems. As the chaos becomes stronger, the correlation time scale of the interaction Hamiltonian  $\hat{V}$ , which is evolved by the unperturbed Hamiltonian  $\hat{H}_0$  in the interaction picture, becomes shorter. Hence, due to the *dynamical averaging*, the influence of  $\hat{V}$  is effectively reduced. Consequently, the entanglement production rate is also reduced. In particular, when the chaos is strong enough, the effect of the perturbation on  $\Gamma^{\text{PT}}$ , Eq. (6), comes only from the “diagonal” part  $D(n, n)$ . This is the origin of the saturation at large  $k$  (Fig. 4).

We make a brief remark on our result, in correspondence with the existing publications which suggest that the entanglement production rate is proportional to the Lyapunov exponent of the classical counterpart (i.e., chaos promotes quantum entanglement) [9,14]. First, although our model system in the numerical experiments is the same as Miller and Sarkar’s work [9], the result is qualitatively different. The difference comes from the different “strength” of chaos.

In contrast with the strongly chaotic systems, the entanglement productions of the weakly chaotic systems are significantly influenced by the existence of tori. A crudest explanation is provided by the perturbation formula (6): While the chaos is not fully developed, a large portion of the phase space is occupied by tori. This reduces  $D_0$ , which is the magnitude of the fluctuation of the interaction  $\hat{V}$ . Accordingly, as the chaos become stronger in a weakly chaotic region, the development of chaos increases  $D_0$ . This results in the increment of the entanglement production rate. In the strongly chaotic region, the growth of  $D_0$  saturates due to the breakdown of tori and  $\gamma$  takes a major role in the parametric variation of  $\Gamma^{\text{PT}}$ , Eq. (6). Second, we compare our result with Zurek and Paz's work on open systems [14], since each subsystem in our numerical experiment acts as a heat bath for its partner. The most important difference comes from the fact that the two works focus on completely different regions, far before (Zurek and Paz) and far after (ours) the Ehrenfest time (see, Fig. 2).

We expect that our result on the saturation of entanglement between weakly coupled systems is generic in strongly chaotic systems with a "compact" phase space that allows the assumption (5). However, we note that our result (6)

heavily depends on the fact that the system is a discrete-time system, i.e., a mapping system.

Finally, we discuss an extension of our result to flow systems. It is straightforward to obtain a perturbative estimation of the entanglement production rate  $\Gamma \propto 1/\gamma$ , which implies the *suppression* of the entanglement productions in the strongly chaotic limit  $\gamma \rightarrow \infty$ . This is completely opposite to the case in the weakly chaotic region [6–9]. More thorough investigations on this point will be reported in subsequent publications. We remark that a similar suppression of quantum relaxation due to strong chaos is reported by Prosen [23] in perturbative evaluations of fidelity, which is an overlapping integral between the two states evolved by slightly different Hamiltonians. As is discussed in Ref. [24], it is hopeful that the suppression of quantum relaxations due to strongly chaotic dynamics will have various applications. In particular, our scenario, which suggests an approach of the *dynamical inhibition of entanglement*, will also provide important applications to quantum communications and computations, which require protection against decoherence [3,24].

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- [1] A. Einstein, B. Podolsky, and N. Rosen, *Phys. Rev.* **47**, 777 (1935); J.S. Bell, *Physics* (Long Island City, N.Y.) **1**, 195 (1964).
- [2] E. Schrödinger, *Proc. Cambridge Philos. Soc.* **31**, 555 (1935).
- [3] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
- [4] D. Giulini *et al.*, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer, Berlin, 1996); W.H. Zurek, e-print quant-ph/0105127, and references therein.
- [5] O. Kübler and H.D. Zeh, *Ann. Phys. (N.Y.)* **76**, 405 (1973).
- [6] S. Adachi, in *Proceedings of ISKIT '92*, edited by I. Tsuda and K. Takahashi (ISIP, Iizuka, 1992), p. 76.
- [7] A. Tanaka, *J. Phys. A Math. Gen.* **29**, 5475 (1996).
- [8] M. Sakagami, H. Kubotani, and T. Okamura, *Prog. Theor. Phys.* **95**, 703 (1996); K. Furuya, M.C. Nemes, and G.Q. Pellegrino, *Phys. Rev. Lett.* **80**, 5524 (1998).
- [9] P.A. Miller and S. Sarkar, *Phys. Rev. E* **60**, 1542 (1999).
- [10] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).
- [11] Exceptions for classically regular systems are found in Refs. [7,12].
- [12] R.M. Angelo *et al.*, *Phys. Rev. E* **60**, 5407 (1999).
- [13] G. Casati *et al.*, *Lect. Notes Phys.* **93**, 334 (1979).
- [14] W.H. Zurek, and J.P. Paz, *Phys. Rev. Lett.* **72**, 2508 (1994).
- [15] F. Haake, M. Kuś, and R. Scharf, *Z. Phys. B: Condens. Matter* **65**, 381 (1987).
- [16] S.M. Barnett and S.J.D. Phoenix, *Phys. Rev. A* **40**, 2404 (1989).
- [17] J.R. Klauder and B.-S. Skagerstam, *Coherent States* (World Scientific, Singapore, 1985).
- [18] K. Husimi, *Proc. Phys. Math. Soc. Jpn.* **22**, 264 (1940); K. Takahashi and N. Saitô, *Phys. Rev. Lett.* **55**, 645 (1985); K. Nakamura *et al.*, *ibid.* **57**, 5 (1986).
- [19] M.V. Berry and N.L. Balazs, *J. Phys. A* **12**, 625 (1979).
- [20] S. Adachi, M. Toda, and K. Ikeda, *Phys. Rev. Lett.* **61**, 659 (1988).
- [21] H. Fujisaki, T. Miyadera, and A. Tanaka (unpublished).
- [22] See, e.g., R. Kubo, M. Toda, and N. Hashitsume, *Statistical Physics II* (Springer-Verlag, Berlin, 1985), Sec. 3.1.
- [23] T. Prosen, *Phys. Rev. E* **65**, 036208 (2002); T. Prosen and M. Žnidarič, *J. Phys. A* **35**, 1455 (2002).
- [24] T. Prosen and M. Žnidarič, *J. Phys. A* **34**, L681 (2002).